TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 1 2016

Section 1 Calculator Free

Complex Numbers

SOLUTIONS

STUDENT'S NAME: ______________________________

DATE: Thursday 5th November **TIME:** 25 minutes **MARKS**: 30

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet.

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (6 marks)

Hint: $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is in Quadrant II.*

Convert $\frac{-1+i\sqrt{3}}{2}$ to polar form and hence evaluate $\left(\frac{-1+i\sqrt{3}}{2}\right)^8$, giving your result in Cartesian form $a+bi$.

$-\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)} \operatorname{cis} \frac{2\pi}{3}$ ✓

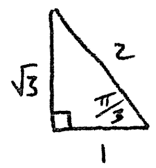
$= \sqrt{\frac{1}{4} + \frac{3}{4}} \operatorname{cis} \frac{2\pi}{3}$

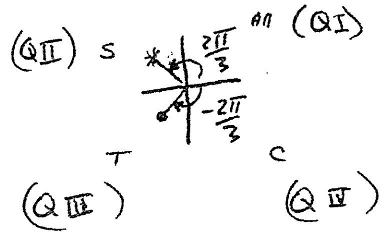
$= \underline{\underline{\operatorname{cis} \frac{2\pi}{3}}}$ ✓

$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)$

$= \tan^{-1}(-\sqrt{3})$

$= \underline{\underline{\frac{2\pi}{3}}}$ ✓





$\therefore \left(\frac{-1+i\sqrt{3}}{2}\right)^8 = \left(\operatorname{cis}\frac{2\pi}{3}\right)^8$

$= \operatorname{cis}\frac{16\pi}{3}$ deMoivre. ✓

$= \operatorname{cis}\frac{4\pi}{3}$ ie. $\frac{16\pi}{3} - 2(2\pi) = \frac{16\pi}{3} - \frac{12\pi}{3} = \frac{4\pi}{3}$

$= \operatorname{cis}\left(-\frac{2\pi}{3}\right)$ ✓

$= \underline{\underline{-\frac{1}{2} - \frac{\sqrt{3}}{2} i}}$ ✓

(In QIII)

2. (8 marks)

Solve the following equations:

(a) $3z^2+3z+1=0$ [3]

Tip: Don't divide both sides by 3, leads to fractions and then having to rationalise denominators

$$\Rightarrow z = \frac{-(3) \pm \sqrt{(3)^2 - 4(3)(1)}}{2(3)} \checkmark$$

$$= \frac{-3 \pm \sqrt{-3}}{6} \checkmark$$

Beware: of common denominator

$$= \underline{\underline{-\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{6} i}} \checkmark$$

Eyes wide open!
Quadratic Formula at the top of page 5 on formula sheet.

(b) * $5z^3-12z^2+5z-2=0$ [5]

Let $P(z) = 5z^3 - 12z^2 + 5z - 2$

$$P(2) = 5(2)^3 - 12(2)^2 + 5(2) - 2$$
$$= 40 - 48 + 10 - 2$$
$$= 0$$

$\therefore$ $(z-2)$ is a factor. ✓

$$\begin{array}{r} \# \\ 5z^2 - 2z + 1 \\ z-2 \,\big)\, \overline{5z^3 - 12z^2 + 5z - 2} \\ 5z^3 - 10z^2 \\ \hline -2z^2 + 5z \\ -2z^2 + 4z \\ \hline z - 2 \\ z - 2 \\ \hline 0 \end{array} \checkmark$$

as expected. ✓

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(1)}}{2(5)} \checkmark$$

$$= \frac{2 \pm \sqrt{-16}}{10}$$

$$= \frac{2 \pm 4i}{10}$$

$$= \frac{1 \pm 2i}{5}$$

'Another' alternative method to obtain #
$5z^2(z-2) - 2z(z-2) + 1(z-2) = LHS$ *
Matt De Cinque

Alternative method to obtain #:
$(z-2)(az^2+bz+c) = LHS$ *
Equate coefficients:
$\underline{a=5}$
$-12 = -2a + b$
$\therefore \underline{b=-2}$
$\underline{\underline{c=1}}$
Colin Ropkoch

$\therefore$ Solutions are: $\underline{\underline{2, \ \frac{1+2i}{5}, \ \frac{1-2i}{5}}}$ ✓

3. (8 marks)

Solve the following equations, stating the roots in polar form and showing them on an Argand diagram:

(a) $z^6 = 1$ [4]

$z_0 = \text{cis}\,0 = 1$ ✓

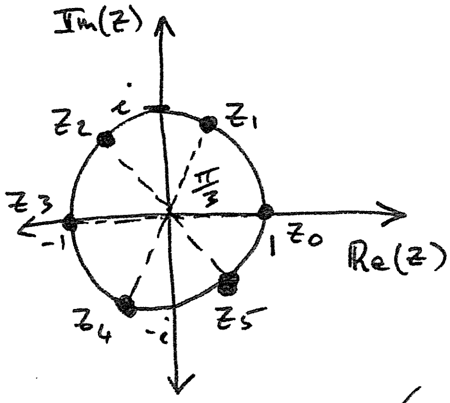
$z_1 = \text{cis}\,\frac{\pi}{3}$

$z_2 = \text{cis}\,\frac{2\pi}{3}$

$z_3 = \text{cis}\,\pi = -1$

$z_4 = \text{cis}\left(\frac{-2\pi}{3}\right)$

$z_5 = \text{cis}\left(\frac{-\pi}{3}\right)$ ✓



✓✓

(b) $z^3 - 64i = 0$ [4]

$\Rightarrow z^3 = 64i$

$\Rightarrow z = (64i)^{\frac{1}{3}}$ ✓

$= \left(64\,\text{cis}\,\frac{\pi}{2}\right)^{\frac{1}{3}}$

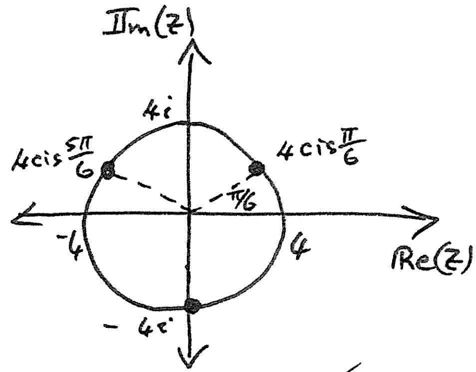
$= 64^{\frac{1}{3}}\,\text{cis}\left(\frac{\pi + 2\pi k}{6}\right)$ ✓ $k = 0, 1,$

$= 4\,\text{cis}\,\frac{\pi}{6}$ or $4\,\text{cis}\,\frac{5\pi}{6}$

or $4\,\text{cis}\left(\frac{-\pi}{2}\right)$

$= -4i$

✓



✓

Use: $z^{\frac{1}{q}} = |z|^{\frac{1}{q}}\,\text{cis}\left(\frac{\theta + 2\pi k}{q}\right)$; $k \in \mathbb{Z}$

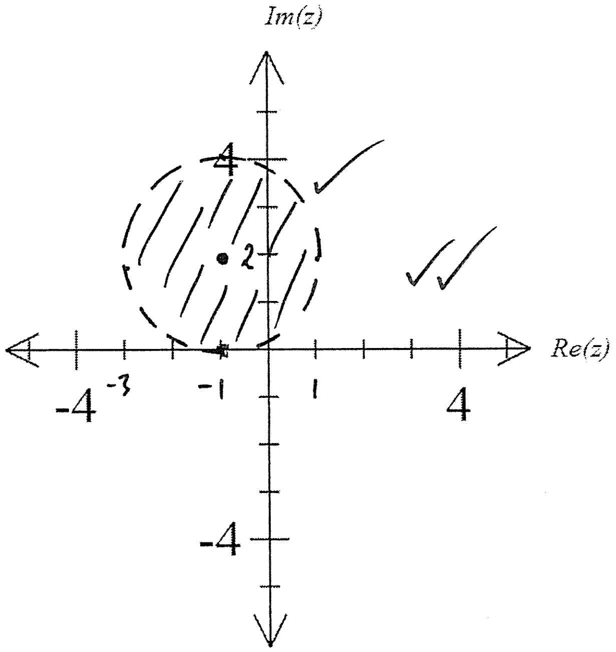
on Pg 6 of Formula Sheet.

4. (8 marks)

Given that $a = 3 + 2i$ and $b = -1 + 2i$. Clearly label the set of points on each Argand diagram defined by:

(a) $|z - b| < 2$ [4]

$\Rightarrow |z - (-1+2i)| < 2$ ✓



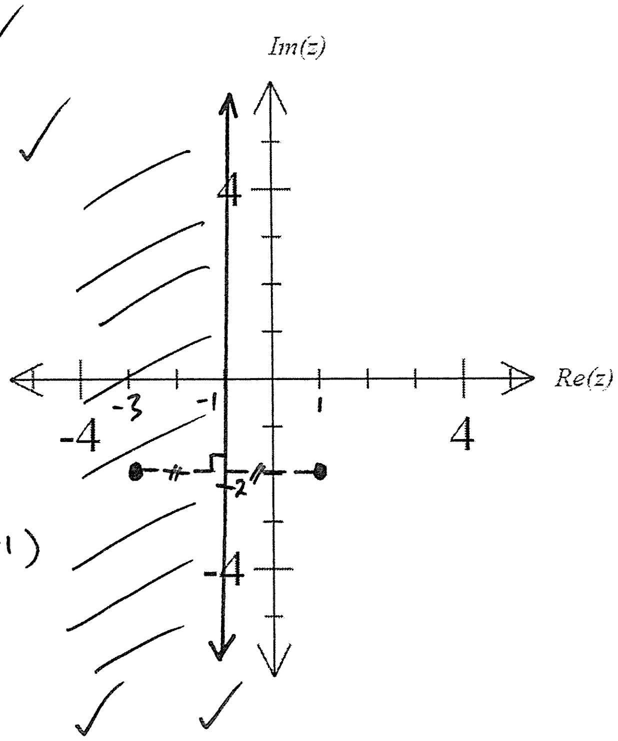
(b) $|z + a| \le |z + b|$ [4]

$\Rightarrow |z + (3+2i)| \le |z + (-1+2i)|$ ✓

$\Rightarrow |z - (-3-2i)| \le |z - (1-2i)|$ ✓

ie the distance of any z from point $(-3,-2)$ is always less than or equal to its distance from point $(1,-2)$.

(equal when on the line $Re(z) = -1$)



End of Questions

TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 1 2016

Section 2 Calculator Assumed ← Hint: Place your ClassPad in Standard Cplx Rad

Complex Numbers

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DATE: Thursday 5th November **TIME:** 25 minutes **MARKS**: 30

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet retained from Section 1.

Special Items: Drawing instruments, templates, three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment).

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (5 marks) See Hint above!

Use your calculator to:

(a) Convert $\frac{-1+i\sqrt{3}}{2}$ to polar form. [1]

$\text{cis}\,\frac{2\pi}{3}$ ✓

ClassPad: e.g. CompToTrig in Interactive Complex.

(b) Evaluate $\left(\frac{-1+i\sqrt{3}}{2}\right)^8$, giving your result in Cartesian form $a+bi$. [2]

$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ ✓✓

ClassPad: e.g. Raise to power of 8 then simplify in Interactive Transformation.

(c) Solve $5z^3-12z^2+5z-2=0$ [2]

$z = 2,\ \frac{1+2i}{5},\ \frac{1-2i}{5}$

✓✓

ClassPad: e.g. Use Solve in Interactive Advanced.

6. (10 marks)

(a) Are the following statements True or False? [4]

(i) $\text{cis}(\pi) = -1$ TRUE

(ii) $\arg(z^{-1}) = -\arg(z)$ TRUE

(iii) $|z^n| = |z|^n \quad \forall n \in \mathbb{Z}$ TRUE

(iv) $(\text{cis}\,\theta)^n = \text{cis}(n\theta) \quad \forall n \in \mathbb{Z}$ TRUE

(b) State the conditions under which the following statements are true: [3]

(i) If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ then $x_1 = x_2$ and $y_1 = y_2$

when $\underline{\underline{z_1 = z_2}}$ ✓

i.e. True $\forall x, y \in \mathbb{R}$. i.e. Cartesian Form unique

(ii) If $z_1 = r_1 \text{cis}\,\theta_1$ and $z_2 = r_2 \text{cis}\,\theta_2$ then $r_1 = r_2$ and $\theta_1 = \theta_2$

when $\underline{\underline{z_1 = z_2}}$ ✓ Beware the converse is not true as Polar Form is not unique

hence the usual restrictions: $r \geq 0$, $-\pi < \theta \leq \pi$ to ensure unique (r, θ).

(iii) $z^{-1} = \bar{z}$

True

when $|z| = 1$ ✓

(c) Given $z = |z|\text{cis}\,\theta$, prove $z\bar{z}$ is always purely real. [3]

Proof: $z\bar{z} = |z|\text{cis}\,\theta \,.\, |z|\,\text{cis}(-\theta)$ ✓

$= |z|^2 \text{cis}(\theta - \theta)$

$= |z|^2 \text{cis}\, 0$ ✓ where $\text{cis}\,0 = 1$

$= |z|^2$

which is purely real.

QED.

✓

7. (8 marks)

Alternative approaches.

(a) Prove for $z \in \mathbb{C}$, $z^{-1} = \dfrac{\bar{z}}{|z|^2}$ [4]

Proof:

Take RHS. where $z = x + yi$

$$= \frac{\bar{z}}{|z|^2}$$

$$= \frac{x - yi}{x^2 + y^2}$$

$$= \frac{x - yi}{(x+yi)(x-yi)}$$

$$= \frac{1}{x+yi}$$

$$= z^{-1}$$

= LHS. QED.

Proof:

Take LHS. $= z^{-1}$ where $z = |z| \operatorname{cis}\theta$

$$= (|z| \operatorname{cis}\theta)^{-1}$$

$$= \frac{1}{|z|} \operatorname{cis}(-\theta)$$

$$= \frac{|z|}{|z|^2} \operatorname{cis}(-\theta)$$

$$= \frac{|z| \operatorname{cis}(-\theta)}{|z|^2}$$

$$= \frac{\bar{z}}{|z|^2}$$

= RHS. QED

Proof:

Take RHS.

$$= \frac{\bar{z}}{|z|^2} \checkmark$$

$$= \frac{\bar{z}}{z\bar{z}} \checkmark$$

$$= \frac{1}{z} \checkmark$$

$$= z^{-1} \checkmark$$

= LHS.

QED.

(b) If $z = \operatorname{cis}\theta$, simplify $z - \dfrac{1}{z}$. [4]

$$= z - z^{-1} \checkmark$$

$$= \operatorname{cis}\theta - \operatorname{cis}(-\theta) \checkmark$$

$$= \cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta)) \checkmark$$

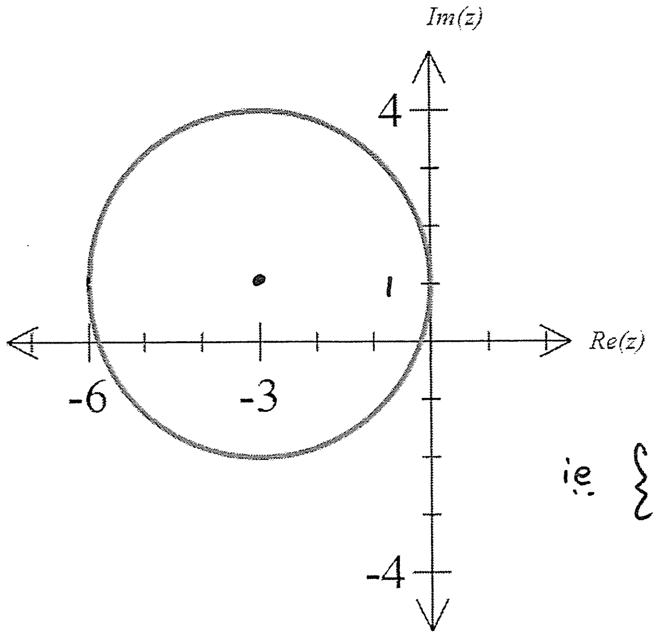
$$= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$$

$$= \underline{\underline{2i\sin\theta}} \checkmark$$

8. (7 marks)

Describe, using appropriate notation, the following sets of points:

(a) [3]

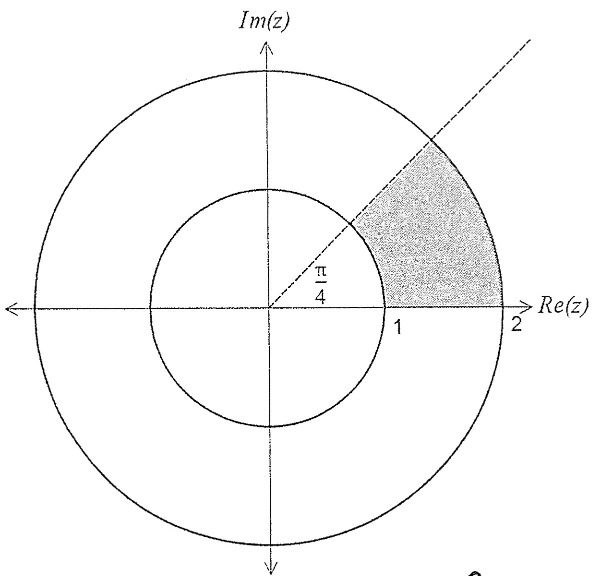


$|z-(-3+i)|=3$

$\Rightarrow |z+3-i| = 3$

ie $\{z : |z+3-i| = 3 ;\ z \in \mathbb{C}\}$

(b) [4]



$\{z : 0 \leq \arg(z) < \frac{\pi}{4};\ z \in \mathbb{C}\} \cap \{z : 1 \leq |z| \leq 2;\ z \in \mathbb{C}\}$

and

End of Questions